

# Universal breaking point asymptotic for energy spectrum of Riemann waves in weakly nonlinear non-dispersive media

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In this Letter we study the form of the energy spectrum of Riemann waves in weakly nonlinear non-dispersive media. For quadratic and cubic nonlinearity we demonstrate that the deformation of an Riemann wave over time yields an exponential energy spectrum which turns into power law asymptotic with the slope being approximately  $-8/3$  at the last stage of evolution before breaking. We argue, that this is the universal asymptotic behaviour of Riemann waves in any nonlinear non-dispersive medium at the point of breaking. The results reported in this Letter can be used in various non-dispersive media, e.g. magneto-hydro dynamics, physical oceanography, nonlinear acoustics.

## I. INTRODUCTION

Weakly nonlinear wave systems fall into two categories - dispersive and non-dispersive. In physical terms, for a dispersive wave system group velocity changes with frequency, for a non-dispersive wave system it does not. Writing the wave in the form of a sine wave  $A \exp[i(kx - \omega t)]$  we can introduce the notion of dispersion function  $\omega(k)$ ,  $\omega(k)$  being a real function, and define dispersive systems as  $\omega_k'' \neq 0$  and non-dispersive systems as  $\omega_k'' = 0$ , [1].

One of the most important characteristics of a wave system, describing the wave field as a whole, is the distribution of energy over scales in Fourier space, i.e. the energy spectrum.

For one initially excited *dispersive* wave with a dispersion function of the form  $\omega(k) \sim k^\beta$  it is known, that the energy spectrum evolves from its initially exponential shape  $e^{\gamma k}$ , [2, 3], into a power law  $k^a$  [3, 4], the point of transition from exponential to power law and the power law itself depending on the characteristic time and the dispersion function of the wave system, [5, 6].

The power law "tail" of the energy spectra can be described deterministically, [3] (in the wave systems with narrow frequency band excitation), or statistically, [4] (in the wave systems with distributed initial state). Basic model for describing a dispersive weakly nonlinear system is the nonlinear Schrödinger equation (NLS) and NLS-like equations modified to four and more nonlinear terms, e.g. [7, 8].

The class of the equations of the form

$$u_t + (\alpha u + \beta u^2)u_x + \gamma u_{xxx} = 0 \quad (1)$$

known as Korteweg-de Vries-like models (KdV-like) widely used in soliton theory [9], dispersionless shock

waves [10–12] and soliton turbulence, e.g. [13–15]. Famous nonlinear Schrödinger equation (NLS) and NLS-like models can be obtained from (1) for narrow-band processes; accordingly the results obtained in [1, 3, 5–8] are also valid for KdV-models.

The dynamics of a nonlinear *non-dispersive* wave follow a simple model: one initially excited wave evolves into a shock wave and finally breaks, [1]. The evolution of an unidirectional nonlinear wave before breaking is described by one nonlinear equation, sometimes called the simple wave equation [1, 16], which reads

$$u_t + V(u)u_x = 0, \quad (2)$$

where  $u$  is the wave function and  $V(u)$  is a nonlinear local speed,  $x$  is coordinate and  $t$  is time. For example, if we regard surface gravity waves in shallow basin,  $u$  is the water elevation and  $V(u) = 3\sqrt{g(h+u)} - 2\sqrt{gh}$ , where  $g$  is acceleration due to gravity,  $h$  is unperturbed water depth [17, 18]. A solution of (2) is called a Riemann wave.

Compare (1) and (2) we see that (2) can be regarded as a dispersionless limit of (1) and the nonlinearity  $V(u)$  being approximated by two terms of its Taylor expansion:  $V(u) = \alpha u + \beta u^2$  (any constant in the Taylor expansion of  $V(u)$  can be removed by an appropriate change of variables).

In this Letter we put forward a few important novel questions about the form of energy spectra in non-dispersive media: Have energy spectra in dispersive and non-dispersive media similar shape? What form of energy spectra have Riemann waves? Does it depend on the type of nonlinearity? What form has spectrum asymptotic before breaking? and others. To answer these questions we use (2) as the simplest possible mathematical model for studying non-dispersive media.

## II. DEFORMATION OF A RIEMANN WAVE BEFORE BREAKING

The time evolution of a Riemann wave may be described as a nonlinear deformation of the wave shape

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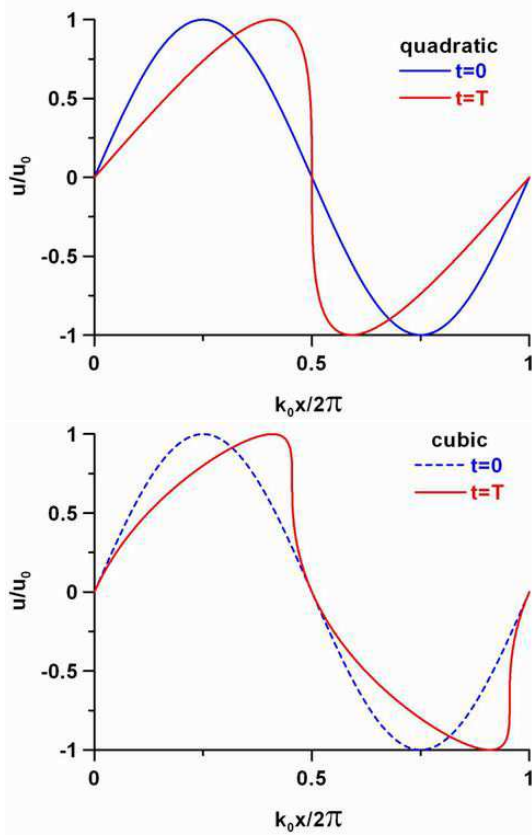


FIG. 1: Color online. Deformation of the Riemann wave  $U(x) = U_0 \sin(k_0 x)$ . **Upper panel:** Quadratic nonlinearity,  $\alpha \neq 0, \beta = 0$ ; the wave shape is shown for  $t = 0$  and for  $t = T = (\alpha k_0 U_0)^{-1}$ . **Lower panel:** Cubic nonlinearity,  $\alpha = 0, \beta \neq 0$ ; the wave shape is shown for  $t = 0$  and for  $t = T = (\beta k_0 U_0^2)^{-1}$ . In both panels, initial shape is shown in blue and the critical shape (at the moment of breaking) is shown in red.

over time, which leads to progressive growth of the wave steepness at one or more points of the wave period defined by wave height which eventually leads to wave breaking. This may be seen the following way.

Solutions of (2) have the property, that any point of a given wave height  $u(x)$  moves at constant speed. So we may write

$$u(x, t) = U[x - V(u)t], \quad (3)$$

where  $U(x)$  is the initial wave profile. Now computing

$$u_x(x, t) = \frac{U_x}{(1 + tV_x)}, \quad (4)$$

we see that steepness  $u_x$  is growing with time if  $V_x < 0$ .

In a general hyperbolic model, if a wave reaches infinite steepness, wave breaking will occur. This is called a gradient catastrophe. The time of breaking  $T$  may be computed as

$$T = 1 / \max(-V_x). \quad (5)$$

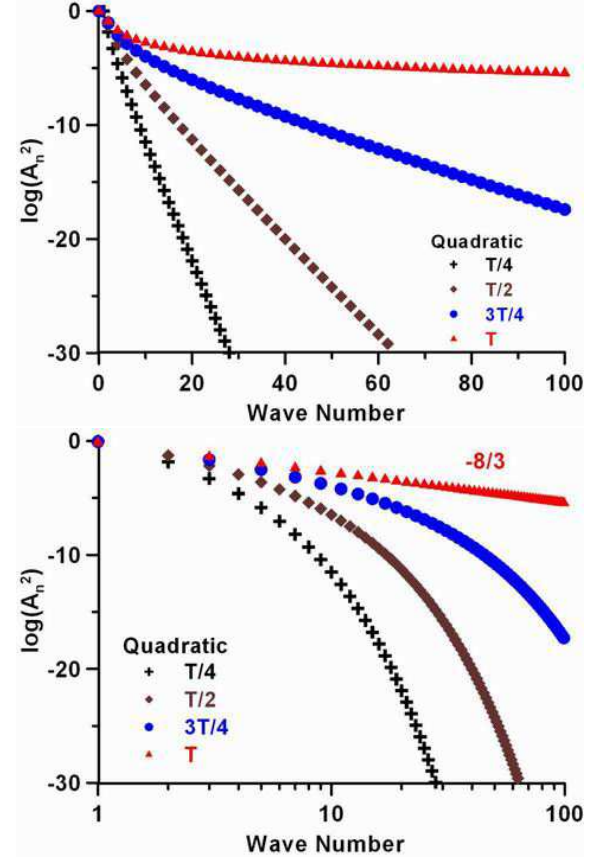


FIG. 2: Color online. Energy spectrum of Riemann waves in quadratic nonlinear medium at different moments of time  $t = 0.25T, 0.5T, 0.75T, T$ , shown in semi-logarithmic coordinates (**upper panel**) and in logarithmic coordinates (**lower panel**)

It follows from (4) that the maximum value of wave steepness is proportional to

$$\max(u_x) \sim (T - t)^{-1} \quad (6)$$

The result of the wave deformation process depends on the local speed  $V(u)$  which in turn depends on the form of nonlinearity. This is illustrated in Fig. 1 for quadratic and cubic nonlinearity. In both panels, the initial wave has the shape of a sine

$$U(x) = U_0 \sin(k_0 x). \quad (7)$$

In the quadratic nonlinear media only one shock is formed within a wave period and media breaking occurs for wave height  $u/U_0 = 0$  at the moment of time  $T = (\alpha k_0 U_0)^{-1}$ .

In the cubic nonlinear media two shocks within the wave period are formed, with opposite sign of slope, and breaking occurs for wave heights  $u/U_0 = \pm \sqrt{2}/2$  simultaneously at the moment of time  $T = (\beta k_0 U_0^2)^{-1}$ .

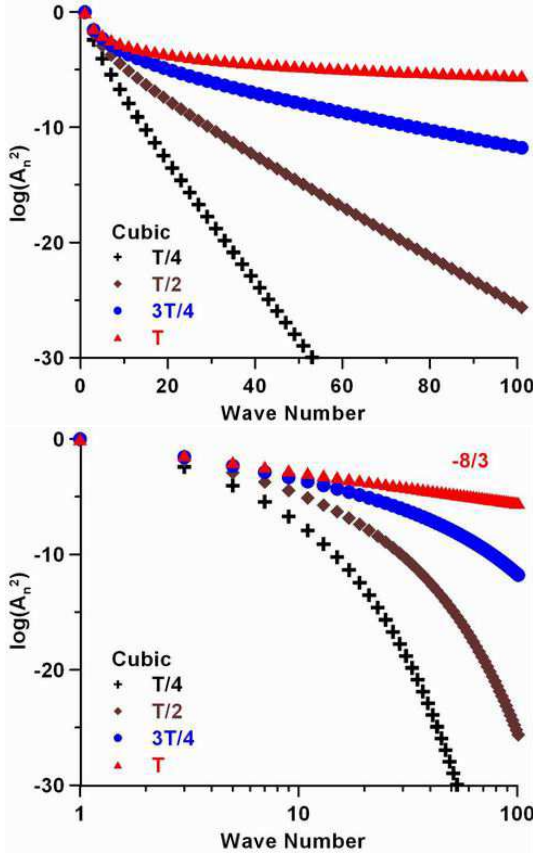


FIG. 3: Color online. Energy spectrum of Riemann waves in cubic nonlinear medium at different moments of time  $t = 0.25T, 0.5T, 0.75T, T$ , shown in semi-logarithmic coordinates (**upper panel**) and in logarithmic coordinates (**lower panel**)

### III. FOURIER SPECTRA OF RIEMANN WAVES

The spatial spectrum of a wave  $u(x, t)$  is defined as

$$S(k, t) = \int_{-\infty}^{+\infty} u(x, t) \exp(-ikx) dx. \quad (8)$$

Next we compute the spatial spectrum of a Riemann wave in explicit form.

Having in mind (4) we perform the change of variables  $X = x - V(u)t$  yielding

$$dx = \frac{dX}{1 - tV_x}. \quad (9)$$

Now we can rewrite (8) as

$$S(k, t) = \int_{-\infty}^{+\infty} (1 + V_X) U(X) \exp[-ik(X + Vt)] dX. \quad (10)$$

By simple manipulation, this is transformed to

$$S(k, t) = (-ik)^{-1} \int_{-\infty}^{+\infty} U_X \exp[-ik(X + tV)] dX. \quad (11)$$

For one initially excited sine wave as given by (7), the integral in (11) may be computed analytically, both for quadratic and cubic nonlinearity, [22].

Using (1) with  $\alpha > 0, \beta = 0$  we get *quadratic nonlinearity*, and the wave field may be represented by the Bessel-Fubini series well-known in nonlinear acoustics, [22]:

$$u(x, t) = U_0 \sum_{n=1}^{\infty} A_n(\tau) \sin[nk_0 x], \quad (12)$$

$$\text{where } A_n(\tau) = \frac{2(-1)^{n+1}}{n\tau} J_n(n\tau), \quad \tau = t/T, \quad (13)$$

with  $T = (\alpha U_0 k_0)^{-1}$  being the time of breaking, and  $J_n(z)$  being the Bessel function.

Fig.2 depicts the time evolution of the energy spectrum given as values  $A_n^2$  as a function of wave number  $k_n$  showing clearly: Up to the moment of time  $t = 3T/4$  the spectrum in semi-logarithmic coordinates (upper panel) has the form of a straight line, which means it is exponential in linear coordinates. As time grows from  $t = 3T/4$  to the moment of breaking  $t = T$ , a power law spectrum is formed, with a slope of 2.654, which is close to 8/3.

Using (1) with  $\alpha = 0, \beta > 0$  we get *cubic nonlinearity*. The wave field may be represented by a Bessel-Fubini-like series, [22], of the form

$$u(x, t) = U_0 \sum_{n=0}^{\infty} \frac{1}{2n+1} \{ J_n[(n+1/2)\tau] \sin[(2n+1)(k_0 x - \tau/2)] + J_{n+1}[(n+1/2)\tau] \cos[(2n+1)(k_0 x - \tau/2)] \}, \quad (14)$$

where again  $\tau = t/T$  and the time of breaking now is given by  $T = (\beta k_0 U_0^2)^{-1}$ . Unlike the quadratic case where all Fourier harmonics are in phase in the cubic case we also have a phase shift of the Fourier harmonics, and the normalized amplitudes are

$$A_n = \sqrt{J_n^2[(n+1/2)\tau] + J_{n+1}^2[(n+1/2)\tau]}. \quad (15)$$

The energy spectrum is shown in Fig.3. As in the case of a quadratic nonlinearity, it has exponential shape for small times (upper panel) and turns into a power law when time approaches the moment of breaking,  $t \rightarrow T$ . The slope of the power law is 2.72 and again close to 8/3.

It is clearly seen from Fig.4 (upper panel) that the energy spectrum has the exponential part, both for quadratic and cubic nonlinear media. The power law part of the energy spectrum is shown in Fig.4, lower panel, for quadratic and cubic nonlinear media; the shape of the energy spectrum is practically undistinguishable for these two cases.

### IV. DISCUSSION AND CONCLUSIONS

The results presented in this Letter can be summarized as follows.

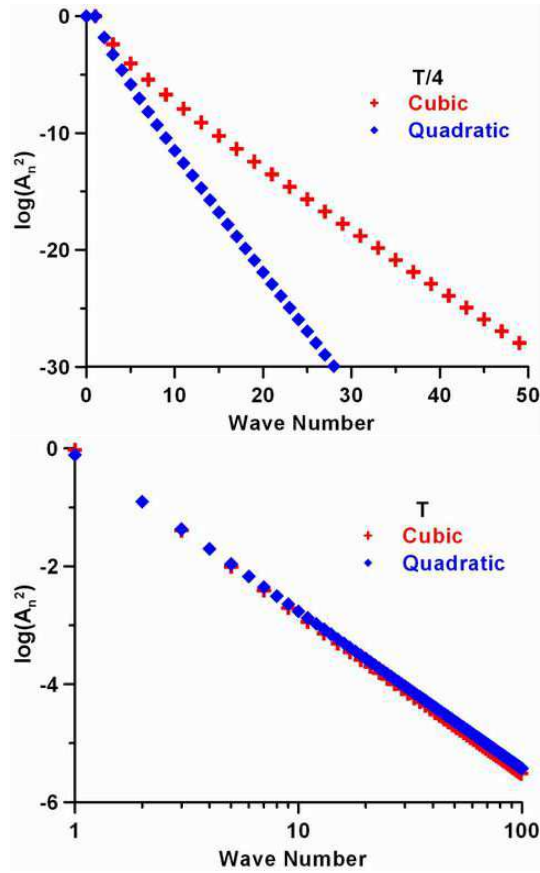


FIG. 4: Color online. **Upper panel:** Exponential "part" of spectrum in quadratic and cubic media, shown by blue crosses and red circles respectively. **Lower panel:** Power law "part" of spectrum in quadratic and cubic media, color scheme as above.

– We have found approximately the same power law of  $\approx k^{-8/3}$  for the energy spectrum of Riemann waves before breaking in quadratic as well as in cubic media. As quadratic and cubic nonlinearity are described by the first and second term of the Taylor expansion, respectively, this indicates, that  $k^{-8/3}$  is the universal power law for Riemann waves with any nonlinearity. What has to be shown for a full proof of this hypothesis is, that the power law applies also for any linear combination of quadratic and cubic nonlinearity.

The Fourier spectrum of the power function  $x^q$  is  $k^{-(1+q)}$ , and the corresponding energy spectrum is  $k^{-2(1+q)}$ . Taking  $q = 1/3$  we get our energy spectrum of  $k^{-8/3}$ . So our hypothesis may be re-stated in the following way: Any point of singularity in the wave profile of a Riemann wave before breaking may be described by a power law of  $x^{1/3}$ .

As in [20, 21] it is assumed, that Riemann waves in media with quadratic nonlinearity before breaking have a wave profile of the form  $x^{1/2}$  in some vicinity of the singularity, yielding a power law for the energy spectrum of  $k^{-3}$ , in Fig.5 both singular profiles are shown for com-

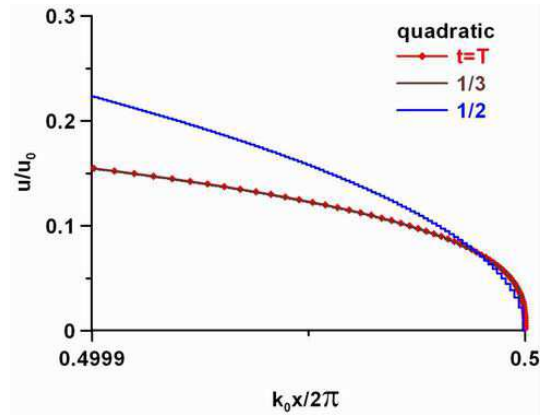


FIG. 5: Color online. The shape of the Riemann wave in a quadratic medium (shown by black continuous line) and two singular profiles:  $x^{1/3}$  and  $x^{1/2}$  shown by red diamonds and zigzag blue curve respectively.

parison.

– The evolution of the wave field after breaking depends on the interplay of dispersion and dissipation.

In essentially dissipative media, the shock wave persists and its amplitude spectrum is known to have the high-frequency asymptotic  $k^{-1}$ , [21, 23].

The dissipation - at least weak dissipation - can be accounted for by including a viscosity term  $u_{xx}$  yielding, for quadratic nonlinear medium, the Burgers equation  $u_t + uu_x - \nu u_{xx} = 0$ , [1]. It can then be reduced to the linear diffusion equation by the Hopf transformation and solved explicitly: a wave damps with a smoother shock. The one-dimensional turbulence within Burgers equation with small viscosity is studied in [19–21].

In weakly dispersive media, solitons are developed in KdV-like models, e.g. [13, 15, 24]. The energy spectrum of soliton turbulence and its change under the action of various statistical parameters have not yet been studied.

– The simple wave equation (2) is found in many problems of nonlinear acoustics, physical oceanography, magnetohydrodynamics and laser optics. The nonlinearity coefficients often vary much stronger than the linear parameters, so the measurement of the modes gives important information about the nature of nonlinearity, see e.g. [25, 26] (acoustical systems) and [27] (optical fiber). The nature of the tail of the spectrum can also be used to determine the type of wave process (smooth form, the beginning of collapse or developed shock).

– In any weakly nonlinear medium where this has been analyzed so far, non-dispersive or dispersive, the time evolution of the energy spectrum goes through the same steps: an initially exponential spectrum asymptotically turns into a power law. So we may assume that this is a general property of these systems; its origin will be the subject of our further study.

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